GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION
SPONSORED PROJECT INITIATION

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Date: October 3, 1977

Project Title: The Use of Benders's Partitioning Scheme for Solving the Quadratic Assignment Problem

Project No: E-24-661

Project Director: Dr. Mokhtar S. Bazaraa

Gr cd

Sponsor: National Science Foundation

Agreement Period: From 9/1/77 Until 2/29/80
(24 month budget period plus 6 month flexibility period)

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Reports Required: Annual Technical Letter Report; Final Technical Report; Summary of Completed Project

Sponsor Contact Person (s):

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Defense Priority Rating: n/a

Assigned to: Industrial and Systems Engineering (School/Laboratory)

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SPONSORED PROJECT TERMINATION

Date: October 19, 1979

Project Title: The Use of Benders's Partitioning Scheme for Solving the Quadratic Assignment Problem

Project No: E-24-661

Project Director: Dr. M. S. Bazaraa

Sponsor: National Science Foundation

Effective Termination Date: 2/29/80

Clearance of Accounting Charges: 2/29/80

Grant/Contract Closeout Actions Remaining:

_ Final Invoice and Closing Documents
x Final ~~Fiscal Report~~ Federal Cash Transactions Report (FCTR)
_ Final Report of Inventions
_ Govt. Property Inventory & Related Certificate
_ Classified Material Certificate
_ Other ____________________

Assigned to: Industrial and Systems Engineering (School/Laboratory)

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CA—4 (1/79)

E-24-661

The Use of Benders' Partitioning Scheme for Solving the Quadratic Assignment Problem

Technical Annual Letter
NSF Grant # ENG 77-07468
Principal Investigator: Mokhtar S. Bazaraa

The study for solving the quadratic assignment problem through the use of the partitioning scheme of Benders addressed the following investigations:

1. A new formulation of the quadratic assignment problem as a linear mixed integer program involving m^2 integer and $m^2(m-1)^2/2$ continuous variables and $2m^2$ linear constraints, where m is the number of facilities.

2. Application of the partitioning method of Benders to decompose the problem and make it more amenable to efficient solution techniques.

3. Development of solution procedures which exploit the structure of the resulting partitioned problem.

4. Specialization of the algorithms developed to solve the traveling salesman problem, which is a special case of the quadratic assignment problem.

5. Computational testing of the developed procedures.

The present status of the research work is at points (4) and (5) above. Some computational testing has already been made. Performing a wider experimentation of the algorithms and specializing them to solve the traveling salesman problem are yet to be conducted. More details on the work already completed is given below.

The formulation provided in the proposal was further refined, resulting in reducing the total number of variables. The proposed method iterates between a master problem and a subproblem. Two closed-form optimal solutions to the subproblem are devised. Due to dual symmetry, we found computationally, depending on the data structure, that one of these solutions yields cuts which perform significantly better than those obtained from the other solution. We have attempted several strategies for solving the master problem, which is a pure 0-1 integer program. The constraints are the cuts generated from the subproblem, in addition to the assignment polytype. As an initial attempt, we tried to solve the problem directly through the use of Gomory's dual-all integer cuts, imbedded in a compact revised simplex array. However, despite finitness of the procedure, we experienced slow convergence. We then resorted to Glover's pseudo-primal-dual cuts which iterate between a dual feasible stage and a primal feasible stage. This resulted in speeding the convergence considerably.

We have also developed a fictitious objective function utilizing the Benders' cuts generated thus far, and solved a linear assignment problem. Since this objective weighs the cuts appropriately, in many cases, the solution to the problem satisfied all the cuts, thus allowing us to avoid the solution of the master problem altogether.

Next we devised a constraint aggregation scheme similar to that of Kendall and Zionts to solve the master problem. The special structure of the problem was exploited in deriving suitable aggregation coefficients. Despite this, the aggregation scheme resulted in coefficient sizes which overflowed even with the use of double precision words.

As another approach, the Benders' partitioning scheme was imbedded in an implicit enumeration procedure. This involved the solution of the master problem only once at the expense of solving the subproblem frequently. This is no deterrent in our case since the subproblems are readily solvable. We attempted three types of lower bounds. The first was based on the solution of linear assignment problems. The second and third types of bounds used a Lagragrian relaxation technique where different constraints are incorporated into the objective function.

To summarize, most of the theoretical and algorithmic development have already been conducted. Furthermore, the algorithms have been coded. Initial testing shows that some of the procedures are able to generate and verify optimality for problems of size m=12 facilities. For larger problems, verification of optimality is not possible within a reasonable computational effort. However, the fortunate feature of the Benders' partitioning method is that an optimal solution, or at least good quality solutions, are provided only after a few number of cuts have been generated. Thus the procedure would at least give good quality solutions early in the process, and can thus be used as a heuristic for solving larger problems. In fact the Benders' partitioning scheme provided better solutions than those known for most of the test problems available in the literature.

THE USE OF BENDERS' PARTITIONING SCHEME FOR SOLVING THE QUADRATIC ASSIGNMENT PROBLEM

Final Report

Presented to

THE NATIONAL SCIENCE FOUNDATION
Grant ≠ ENG77-07468

Principal Investigator

Mokhtar S. Bazaraa

School of Industrial and Systems Engineering
Georgia Institute of Technology

September 1979

TABLE OF CONTENTS

ATTACHMENTS

1. Ph.D. dissertation of Dr. Hanif D. Sherali titled, "The Quadratic Assignment Problem: Exact and Heuristic Methods."

2. Listing of Computer Programs

3. Copies of the following papers:

   a. M. S. Bazaraa and H. D. Sherali, "Benders' Partitioning Scheme Applied to a New Formulation of the Quadratic Assignment Problem," Naval Research Logistics Quarterly, to appear.

   b. M. S. Bazaraa and H. D. Sherali, "Cutting Plane Methods for the Quadratic Assignment Problem," submitted for publication.

   c. M. S. Bazaraa and H. D. Sherali, "A Versatile Scheme for Ranking the Extreme Points of an Assignment Polytope," submitted for publication.

I. SUMMARY

This study is concerned with the development of exact solution procedures for the quadratic assignment problem and deriving heuristics from these methods through premature termination. The problem involves the assignment of m indivisible entites, called facilities, to m mutually exclusive locations. The objective is to minimize a quadratic function which reflects not only the fixed cost of assigning each facility to some location, but also the interaction cost accruing from the location of each facility relative to the location of other facilities.

During the course of this research, the quadratic assignment problem is reformulated into a linear mixed integer program. The new problem has m^2 integer variables, $\frac{m^2(m-1)^2}{2}$ continuous variables, and $2m^2$ linear constraints. Several procedures that take advantage of the special structure of this problem are developed. Then methods can be classified under the decomposition scheme of Benders and branch bound.

The Benders' partitioning method iterates between a subproblem and a master problem. An optimal extreme point solution to the subproblem in closed form is readily available. Several exact and heuristic methods for solving the master problem are developed. These include an implicit enumeration scheme, a constraint aggregation method, and solving a linear assignment problem whose fictitious objective function is obtained by aggregating the Benders' cuts generated thus far. In addition to these procedures, several methods using implicit enumeration and branch-and-bound have been developed for solving both the original problem and the newly formulated equivalent mixed integer program.

Through an extensive computational analysis, we conclude that the Benders' partitioning scheme is able to construct good quality solutions

within a reasonable computational effort, but is unable to verify optimality even for problems of moderate size. The scheme, however, produced the best known solutions for test problems currently available in the literature.

II. DESCRIPTION OF THE RESEARCH ACTIVITY

The quadratic assignment problem involves the assignment of m facilities to m locations in such a way to minimize the total cost. If the flow from facility i to facility k is f_{ik} and the distance from location j to location ℓ is $d_{j\ell}$, the problem can be stated mathematically as follows:

$$\text{minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{\ell=1}^{m} f_{ik} d_{j\ell} x_{ij} x_{k\ell}$$

$$\text{subject to} \quad \sum_{j=1}^{m} x_{ij} = 1 \qquad i = 1,\ldots,m$$

$$\sum_{i=1}^{m} x_{ij} = 1 \qquad j = 1,\ldots,m$$

$$x_{ij} = \begin{cases} 1 & \text{if facility } i \text{ is placed in location } j \\ 0 & \text{otherwise} \end{cases}$$

The following major tasks are performed during the course of this research.

Task 1 (Reformulating the Problem as a Linear Mixed Integer Program)

An equivalent formulation of the problem that transforms the objective function into a linear form is developed. The new problem has discrete variables x_{ij}'s and continuous variables $y_{ijk\ell}$'s and is given below:

$$\text{minimize} \quad \sum_{i=1}^{m-1} \sum_{j=1}^{m} \sum_{k=i+1}^{m} \sum_{\substack{\ell=1 \\ \ell \neq j}}^{m} f_{ik} d_{j\ell} y_{ijk\ell}$$

$$\text{subject to} \quad \sum_{k=i+1}^{m} \sum_{\substack{\ell=1 \\ \ell \neq j}}^{m} y_{ijk\ell} - (m-i)x_{ij} = 0 \qquad \begin{array}{l} i = 1,\ldots,m-1 \\ j = 1,\ldots,m \end{array}$$

$$\sum_{i=1}^{k-1} \sum_{\substack{j=1 \\ j \neq \ell}}^{m} y_{ijk\ell} - (k-1)x_{k\ell} = 0 \qquad \begin{array}{l} k = 2,\ldots,m \\ \ell = 1,\ldots,m \end{array}$$

$$\sum_{j=1}^{m} x_{ij} = 1 \qquad i = 1,\ldots,m$$

$$\sum_{i=1}^{m} x_{ij} = 1 \qquad\qquad j = 1,\ldots,m$$

$$x_{ij} = 0 \text{ or } 1 \qquad\qquad i,j = 1,\ldots,m$$

$$0 \le y_{ij} \le 1 \qquad i = 1,\ldots,m-1 \quad k = i+1,\ldots,m$$

$$\ell,j = 1,\ldots,m \quad \ell \neq j$$

Task 2 (Application of Benders' Partitioning Scheme)

The Benders' partitioning scheme applied to the newly formulated problem iterates between a subproblem and a master problem as follows:

Subproblem

Given a solution x_{ij}^{s} from the master problem at iteration s, the subproblem is to

$$\text{maximize} \quad \sum_{i=1}^{m-1} \sum_{j=1}^{m} (m-i)u_{ij}x_{ij}^{s} + \sum_{k=2}^{m} \sum_{\ell=1}^{m} (k-1)v_{k\ell}x_{k\ell}^{s}$$

$$- \sum_{i=1}^{m-1} \sum_{k=i+1}^{m} \sum_{j=1}^{m} \sum_{\substack{\ell=1 \\ \ell\neq j}}^{m} w_{ijk\ell}$$

$$\text{subject to} \quad u_{ij} + v_{k\ell} - w_{ijk\ell} \le c_{ijk\ell}$$

$$w_{ijk\ell} \ge 0 \qquad i = 1,\ldots,m-1 \qquad k = i+1,\ldots,m$$

$$j,\ell = 1,\ldots,m \qquad \ell\neq j$$

Let $(u_{ij}^{s}, v_{k\ell}^{s}, w_{ijk\ell}^{s})$ be an optimal solution and go to the master problem.

Master Problem

$$\text{minimize} \quad z$$

$$\text{subject to } z \ge \sum_{i=1}^{m-1} \sum_{j=1}^{m} (m-i)u_{ij}^{p}x_{ij} + \sum_{k=2}^{m} \sum_{\ell=1}^{m} (k-1)v_{k\ell}^{p}x_{k\ell}$$

$$- \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=i+1}^{m} \sum_{\substack{\ell=1 \\ \ell\neq j}}^{m} w_{ijk\ell}^{p} \qquad p = 1,\ldots,s$$

$$\sum_{j=1}^{m} x_{ij} = 1 \qquad i = 1,\ldots,m$$

$$\sum_{i=1}^{m} x_{ij} = 1 \qquad j = 1,\ldots,m$$

$$x_{ij} = 0 \text{ or } 1 \qquad i,j = 1,\ldots,m$$

Let x^s be an optimal solution and go to the subproblem.

<u>Task 3</u> (A Closed Form Solution to the Subproblem)

We proved that the following solution is an extreme point optimal optimal solution to the subproblem as u_{ij} and $v_{k\ell}$ are transformed into the difference of nonnegative variables.

$$u_{ij}^s = \begin{cases} \displaystyle\max_{\substack{(k,\ell):\ x_{k\ell}^s = 1 \\ k > i}} f_{ik} d_{j\ell} & \text{if } x_{ij}^s = 1 \\ \\ \displaystyle\min_{(k,\ell):\ k > i} [f_{ik} d_{j\ell} - v_{k\ell}^s] & \text{if } x_{ij}^s = 0 \end{cases} \qquad \begin{array}{l} i = 1,\ldots,m-1 \\ j = 1,\ldots,m \end{array}$$

$$v_{k\ell}^s = \begin{cases} 0 & \text{if } x_{k\ell}^s = 1 \\ \\ \text{minimum} \begin{Bmatrix} \displaystyle\min_{(i,j)} [f_{ik} d_{j\ell} - u_{ij}^s : x_{ij}^s = 1, k > i] \\ \\ \text{minimum } [f_{ik} d_{j\ell} :\ x_{ij}^s = 0, k > i] \end{Bmatrix} & \begin{array}{l} k = 2,\ldots,m \\ \ell = 1,\ldots,m \\ \text{if } x_{k\ell}^s = 0 \end{array} \end{cases}$$

$$w^s_{ijk\ell} = \begin{cases} 0 & \text{if } x^s_{ij}x^s_{k\ell} = 0 \\ & i = 1,\ldots,m-1 \quad k = i+1,\ldots,m \\ & j,\ell = 1,\ldots,m \quad j \neq \ell \\ u^s_{ij} + v^s_{k\ell} - f_{ik}d_{j\ell} & \text{if } x^s_{ij}x^s_{k\ell} = 1 \end{cases}$$

Task 4 (Exact Solutions to the Master Problem)

Three different schemes for solving the master problem exactly are attempted. These are:

1. Specialization of Glovers' pseudo-primal-dual procedure.
2. A cut aggregation scheme that is specialized to handle the structure of the master problem.
3. An implicit enumeration method.

The first two methods excercized computational difficulties and did not converge even for small problems. The third method performed much better, but since the number of Benders' cuts needed for an exact solution is enormous, an exact approach was deemed impractical.

Task 5 (Heuristic Solution to the Master Problem)

A method for aggregating the Benders' cuts together with the Cabot-Francis objective form into a single fictitious objective function is developed. The solution to the resulting linear assignment problem provided very good quality solutions to the quadratic assignment problem only after a few number of cuts have been generated.

Task 6 (Specialization of the Benders' Method to the Traveling Salesman Problem)

The heuristic procedure for solving the quadratic assignment problem is specialized to the traveling salesman problem.

Task 7 (Computational Experimentation)

The methods described above are tested by means of standard problems in the literature and randomly generated problems whose optimal solutions are known. The testing indicates that the heuristic method is quite efficient in the sense that it obtained the best known solutions in the literature and in some cases improved on them. The exact method was unable to verify optimality even for moderate size problems, however.

III. PERSONNEL

This section presents the personnel involved in this research and the contribution they have made.

Principal Investigator

Mokhtar S. Bazaraa
Professor
School of Industrial and Systems Engineering
Georgia Institute of Technology

Contribution
1. Organization and Coordination of the project.
2. Formulating the quadratic assignment problem as a linear mixed integer program.
3. Development of several of the algorithms discussed in Section II.

Graduate Research Assistants

A. Hanif D. Sherali
School of Industrial and Systems Engineering
Georgia Institute of Technology

Contribution
1. Development of the algorithms discussed in Section II.
2. Coding of the algorithms discussed in Section II.
3. Designing and performing a detailed computational analysis of the developed methods.

NOTE: As a result of his involvement in the research effort, Mr. Sherali elected to work on the quadratic assignment problem for his Ph.D. dissertation. In June 1979, Mr. Sherali successfully completed all the requirements to the Ph.D. degree at Georgia Institute of Technology. A copy of the Ph.D. dissertation, which explains in detail all the algorithms developed, is attached.

B. Jai H. Eu
School of Industrial and Systems Engineering
Georgia Institute of Technology

Contribution

1. Specializing the Benders' partitioning algorithm to the traveling salesman problem.

2. Computational testing of the Benders' partitioning heuristic for solving the traveling salesman problem.

IV. LIST OF PUBLICATIONS

1. M. S. Bazaraa and H. D. Sherali, "Benders' Partitioning Scheme Applied to a New Formulation of the Quadratic Assignment Problem," Naval Research Logistics Quarterly, to appear.

2. M. S. Bazaraa and H. D. Sherali, "Cutting Plane Methods for the Quadratic Assignment Problem," submitted for publication.

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4. M. S. Bazaraa and H. D. Sherali, "A New Subgradient Optimization Technique," under preparation.